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# Forced vibrations of plates and cylindrical shells with regular orthogonal system of stiffeners

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A B S T R A C T

A wide range of engineering structures, such as aircraft fuselages or ship hulls have as the foundation a shell orthogonally strengthened by two sets of stiffeners. Solution of the task related to determining the vibrations of such complicated structures requires an application of special methods which permit accounting for the interaction between the shell and the two sets of discrete stiffeners correctly. The present work proposes an effective method of predicting the vibrations of a finite orthogonally stiffened structure as a part of an infinite one when the edge conditions permit. The prediction method proposed is based on the method of space-harmonic expansions when the shell displacements and forces are presented in the form of special double trigonometric series. The method allows the interconnection of all three components of displacement and rotation of the shell and the stiffeners to be taken into account. The vibration velocity of the construction is determined directly without a need for solving the task of eigen-values first. The vibration shapes are broken into a large number of non-interacting groups of shapes. The solution reduces to a system of equations relating to the generalized reactions at supports. All this allows predictions to be made for large parts of the investigated construction over practically the whole frequency range of sound.



## 1. Introduction

A shell or a panel regularly strengthened by stiffeners in two orthogonal directions is the foundation of a wide range of engineering structures and in particular of aircraft fuselages. A great number of works are devoted to vibrations of stiffened plates, shells and to excitation of periodic structures and are covered [1–4] and in recent publications [5–8]. In the majority of published works one-dimensional or quasi one-dimensional systems stiffened in only one direction are considered. Unfortunately, the number of works where stiffening in two directions is investigated and which are of prime practical interests is rather limited.

According to overview [3], analytical methods that can be used for solving the two-dimensional tasks can be separated into three methods: receptance method, transfer-matrices method and the method of space-harmonics. The receptance method is a dynamic-flexibility technique, which allows vibrations of the non-regular stiffened structure to be determined directly and the solution in this case is presented in the form of usual double trigonometric series. This method was used to determine the vibrations and the sound transmission loss of an orthogonally stiffened curved panel excited by sound and pressure-fluctuation fields [6]. However, in the expressions of that work the interaction between the panel and the

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stiffeners is taken into account only through normal displacements. Moreover, prediction results for a frame-stringer stiffened panel had not been demonstrated because of "excessive computational time".

The receptance method is efficient for solving the task for structures with non-uniformly spaced stiffeners. For regular structures it is more reasonable to use special methods, such as the transfer-matrices and space-harmonics methods, that are based on Floquet's principle (sometimes called Bloch's theorem). According to this principle, the eigen-functions of periodic systems can always be presented in the form of a product of some function with the same period as that of the system and the plane harmonic wave function.

The solution for stiffened-structure vibrations, when the structure consists of a number of identical spans or cells, is reduced by the method of transfer matrices to considering only one span or cell. However this solution is usually derived in the form of an implicit connection between propagation constants and frequency and this makes a search for eigen-frequencies and eigen-functions difficult. This method becomes too laborious at high frequencies when it is necessary to account for a large number of vibration eigen-functions. In the present work the method of space-harmonic expansions is used. According to this method, vibrations of the infinite, orthogonally stiffened plate with distances between the stiffeners $d^r$, $d^s$ under plane wave excitation $\exp\,\mathrm{i}(\omega t + \alpha x/d^r + \beta y/d^s)$ are presented in the form of special space-harmonic series [3]

$$w(x, y, t) = \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} W_{mn} \exp\,\mathrm{i}(\omega t + k_{\alpha m} x + k_{\beta n} y), \tag{1}$$

$$k_{\alpha m} = (\alpha + 2\pi m)/d^r, \quad k_{\beta n} = (\beta + 2\pi n)/d^s. \tag{2}$$

Here all the harmonic shapes of vibrations with similar phase steps $\alpha$, $\beta$ on the cell length and width are summed up but the other shapes take no part in vibrations under such excitation.

The present work shows how the system of equations for the unknown amplitude $W_{nm}$ of the plate and the shell can be reduced to a system related to sufficiently small number of generalized stiffener responses. The proposed method is described in the first part of the paper by an example of an infinite thin plate strengthened by stiffeners, elastic in bending and absolutely compliant in twisting. In the second part of the paper this method is used for solving the task on vibrations of a finite orthogonally stiffened cylindrical shell. The method can be used in those cases when the edge conditions for the finite structure allows it to be considered as part of an infinite structure. Here a shell is considered, finite in both directions and simply supported on the edges with the stiffeners split in half.

The method allows all three displacement components and the rotation of the shell and the stiffeners to be taken into account. The equations of the shell and of the stiffeners are written in a general matrix form and this allows an easy use of any equations of the structural element dynamics. The vibration shapes of the shell and stiffeners are broken into non-interacting groups. The solution reduces to an equation system regarding a substantially smaller number of the generalized responses of stiffeners and this makes it possible to account for a very large number of shapes in each group. All this allows a prediction to be made of vibrations of large parts of the structure under investigation over practically the whole sound frequency range.

## 2. Prediction relations for infinite orthogonally stiffened plate

Consider a thin infinite plate regularly strengthened by stringers along the $x$-axis with a step of $d^s$ and by frames with a step of $d^r$ along the $y$-axis (Fig. 1). The stiffeners interact with the plate along orthogonal lines $x = p^r d^r$, $y = p^s d^s$, where $p^r$, $p^s$ are the numbers of frames and stringers.

The plate has normal displacements $w$ which are determined by the following equation:

$$(D(\partial^2/\partial x^2 + \partial^2/\partial y^2)^2 - \omega^2 m)w(x, y) = q - q^s - q^r, \tag{3}$$

where $m$ is the surface mass, $D$ the cylindrical rigidity, $q(x, y)$, $q^s(x, y)$, $q^r(x, y)$ are the external surface forces, stringer and frame response forces, respectively. Here and below time multiplier $\exp(\mathrm{i}\omega t)$ is omitted. Normal displacements of stringer

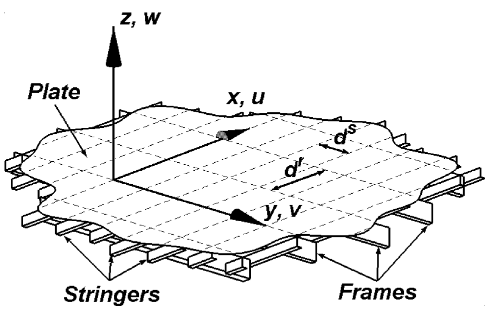


**Fig. 1.** Orthogonally stiffened infinite plate.

$w^s$ with number $p^s$ are described with the use of the following equation:

$$(E^sI^s\partial^4/\partial x^4 - \omega^2 m^s)w^s(x,p^s) = q^{\delta s}(x,p^s), \tag{4}$$

where $m^s$, $E^sI^s$ are the mass per unit length and flexural rigidity of stringers. Upper index $\delta$ serves for distinguishing the discrete force in one direction ($q^{\delta s}(x,p^s)$) from the surface force ($q^s(x,y)$).

The frame displacements $w^r$ with number $p^r$ are described as follows:

$$(E^rI^r\partial^4/\partial y^4 - \omega^2 m^r)w^r(p^r,y) = q^{\delta r}(p^r,y), \tag{5}$$

where $m^r$, $E^rI^r$ are the mass per unit length and flexural rigidity of the frames.

Let the stringer displacements $w^s$ and frame displacements $w^r$ be connected with the plate only through the normal displacements

$$w^s(x,p^s) = w(x,p^sd^s), \quad w^r(p^r,y) = w(p^rd^r,y). \tag{6}$$

Present the displacements in the form of an integral over the phases and of a special double series for each pair of phases

$$w(x,y) = \int_{\alpha=-\pi}^{\pi}\int_{\beta=-\pi}^{\pi}\left[\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} W_{\mathbf{mn}} \exp\, \mathrm{i}(k_{\mathbf{m}}x + k_{\mathbf{n}}y)\right]\mathrm{d}\alpha\,\mathrm{d}\beta,$$

$$k_{\mathbf{m}} = (\alpha + 2\pi m)/d^r, \quad k_{\mathbf{n}} = (\beta + 2\pi n)/d^s, \quad \mathbf{m} \equiv \{\alpha, m\}, \quad \mathbf{n} \equiv \{\beta, n\}. \tag{7}$$

Here $W_{\mathbf{mn}}$ are the generalized plate displacements. The double series in square brackets corresponds to the series in Eq. (1) and defines the contribution of one independent shape group to the overall plate displacement. Expression (7) can be considered as a special kind of Fourier integral representation.

Now the forces acting on plate ($q$–$q^s$–$q^r$) will be presented in a similar way. The generalized forces are divided into external generalized forces $Q_{\mathbf{mn}}$, generalized responses of stringers $-Q^s_{\mathbf{mn}}$ and responses of frames $-Q^r_{\mathbf{mn}}$:

$$q - q^s - q^r = \int_{\alpha=-\pi}^{\pi}\int_{\beta=-\pi}^{\pi}\left[\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}(Q_{\mathbf{mn}} - Q^s_{\mathbf{mn}} - Q^r_{\mathbf{mn}}) \exp\, \mathrm{i}(k_{\mathbf{m}}x + k_{\mathbf{n}}y)\right]\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{8}$$

From the plate vibration Eq. (3) the connection between the generalized plate displacements $W_{\mathbf{mn}}$ and the sum of the generalized forces acting on it follows:

$$K_{\mathbf{mn}}W_{\mathbf{mn}} = Q_{\mathbf{mn}} - Q^r_{\mathbf{mn}} - Q^s_{\mathbf{mn}}, \quad K_{\mathbf{mn}} = D(k^2_{\mathbf{m}} + k^2_{\mathbf{n}})^2 - \omega^2 m. \tag{9}$$

The plate vibration shapes from one group $\alpha$, $\beta$ and with one longitudinal index $m$ have an identical magnitude for all the indices $n$ on stringer lines $y = p^sd^s$:

$$\exp\mathrm{i}(k_{\mathbf{m}}x + k_{\mathbf{n}}p^sd^s) = \exp\mathrm{i}(k_{\mathbf{m}}x + \beta p^s), \quad \forall n. \tag{10}$$

Expand the displacements of all the stringers in terms of functions $\exp\mathrm{i}(k_{\mathbf{m}}x+\beta p^s)$, continuous along $x$ and discrete along $y$:

$$w^s(x,p^s) = \int_{\alpha=-\pi}^{\pi}\int_{\beta=-\pi}^{\pi}\left[\sum_{m=-\infty}^{\infty} W^s_{\beta\mathbf{m}} \exp\, \mathrm{i}(k_{\mathbf{m}}x + \beta p^s)\right]\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{11}$$

Here $W^s_{\beta\mathbf{m}}$ are the generalized stringer amplitudes.

Expand also the forces $q^{\delta s}$, discrete along $y$, that affect the stringers from the plate:

$$q^{\delta s}(x,p^s) = d^s\int_{\alpha=-\pi}^{\pi}\int_{\beta=-\pi}^{\pi}\left[\sum_{m=-\infty}^{\infty} Q^{\delta s}_{\beta\mathbf{m}} \exp\mathrm{i}(k_{\mathbf{m}}x + \beta p^s)\right]\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{12}$$

Here a step of stringer arrangement $d^s$ is added on the right so that the generalized force $Q^{\delta s}_{\beta\mathbf{m}}$ has the dimensionality of surface force.

Along the line of frames the plate vibration shapes of one group also have identical magnitudes independently of longitudinal index $m$:

$$\exp\, \mathrm{i}(k_{\mathbf{m}}p^rd^r + k_{\mathbf{n}}y) = \exp\, \mathrm{i}(\alpha p^r + k_{\mathbf{n}}y), \quad \forall m. \tag{13}$$

The vibrations of all the frames and of the force $q^{\delta r}(p^r,y)$ acting on them from the plate, can be presented as the expansion in terms of $\exp\, \mathrm{i}(\alpha p^r + k_{\mathbf{n}}y)$:

$$w^r(p^r,y) = \int_{\alpha=-\pi}^{\pi}\int_{\beta=-\pi}^{\pi}\left[\sum_{n=-\infty}^{\infty} W^r_{\alpha\mathbf{n}} \exp\, \mathrm{i}(\alpha p^r + k_{\mathbf{n}}y)\right]\mathrm{d}\alpha\,\mathrm{d}\beta, \tag{14}$$

$$q^{\delta r}(p^r,y) = d^r\int_{\alpha=-\pi}^{\pi}\int_{\beta=-\pi}^{\pi}\left[\sum_{n=-\infty}^{\infty} Q^{\delta r}_{\alpha\mathbf{n}} \exp\, \mathrm{i}(\alpha p^r + k_{\mathbf{n}}y)\right]\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{15}$$

From the stringer vibration Eq. (4), the connection between the generalized displacements $W^r_{\alpha\mathbf{n}}$ and the generalized forces $Q^{\delta r}_{\alpha\mathbf{n}}$ follows:

$$K^s_{\mathbf{m}} W^s_{\beta\mathbf{m}} = Q^{\delta s}_{\beta\mathbf{m}}, \quad K^s_{\mathbf{m}} \equiv (E^s I^s k^4_{\mathbf{m}} - \omega^2 m^s)/d^s. \tag{16}$$

Similar equalities can be obtained for the frames:

$$K^r_{\mathbf{n}} W^r_{\alpha\mathbf{n}} = Q^{\delta r}_{\alpha\mathbf{n}}, \quad K^r_{\mathbf{n}} \equiv (E^r I^r k^4_{\mathbf{n}} - \omega^2 m^r)/d^r. \tag{17}$$

It follows from the equality of stiffener vibration shapes and the plate shapes along the stiffeners (Eqs. (6), (10), (13)) that the generalized stringer displacements are equal to the sum of all the generalized plate displacements with the same indices $m, \alpha, \beta$ and the generalized frame displacements are equal to the sum of the generalized plate displacements with the same indices $n, \alpha, \beta$:

$$W^s_{\beta\mathbf{m}} = \sum_{n=-\infty}^{+\infty} W_{\mathbf{mn}}, \quad W^r_{\alpha\mathbf{n}} = \sum_{m=-\infty}^{+\infty} W_{\mathbf{mn}} \quad (\mathbf{m} \equiv \{\alpha, m\}, \mathbf{n} \equiv \{\beta, n\}). \tag{18}$$

Now connect the generalized forces acting on the stiffeners $Q^{\delta s}_{\beta\mathbf{m}}$ and $Q^{\delta r}_{\alpha\mathbf{n}}$ with the generalized stiffeners responses acting on the plate $Q^s_{\mathbf{mn}}, Q^r_{\mathbf{mn}}$. The regular sequence of delta-functions $\delta(y - p^s d^s)$ with the arrangement step $d^s$ and the phase step $\beta$ can be replaced by a sum of harmonic functions with the same phase step:

$$d^s \sum_{p^s=-\infty}^{+\infty} \exp(\mathrm{i}\beta p^s)\delta(y - p^s d^s) = \sum_{n=-\infty}^{+\infty} \exp(\mathrm{i}(\beta + 2\pi n)y/d^s) = \sum_{n=-\infty}^{+\infty} \exp(\mathrm{i}k_{\mathbf{n}} y) \tag{19}$$

Therefore the discrete forces along $y$ ($q^s$), acting on the plate from the stringers can be written as follows (see Eq. (12)):

$$\begin{aligned} q^s(x,y) &= \sum_{p^s=-\infty}^{+\infty} q^{\delta s}(x, p^s)\delta(y - p^s d^s) \\ &= d^s \int_{\alpha=-\pi}^{\pi} \int_{\beta=-\pi}^{\pi} \left[ \sum_{m=-\infty}^{\infty} Q^{\delta s}_{\beta\mathbf{m}} \sum_{p^s=-\infty}^{+\infty} \exp\, \mathrm{i}(k_{\mathbf{m}} x + \beta p^s)\delta(y - p^s d^s) \right] \mathrm{d}\alpha\, \mathrm{d}\beta \\ &= \int_{\alpha=-\pi}^{\pi} \int_{\beta=-\pi}^{\pi} \left[ \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} Q^{\delta s}_{\beta\mathbf{m}} \exp\, \mathrm{i}(k_{\mathbf{m}} x + k_{\mathbf{n}} y) \right] \mathrm{d}\alpha\, \mathrm{d}\beta. \end{aligned} \tag{20}$$

Comparing this expression with Eq. (8) for $q^s$, we are assured that all the generalized stringer response forces $Q^s_{\mathbf{nm}}$ with identical indices $m, \alpha, \beta$ are equal to each other and are equal to the generalized force acting on the stringers $Q^{\delta s}_{\beta\mathbf{m}}$:

$$Q^s_{\mathbf{nm}} = Q^{\delta s}_{\beta\mathbf{m}}, \quad \forall n. \tag{21}$$

Similarly one can prove that the generalized frame response forces $Q^r_{\mathbf{mn}}$ with identical indices $n, \alpha, \beta$ are equal to the generalized force acting on the frames:

$$Q^r_{\mathbf{nm}} = Q^{\delta r}_{\alpha\mathbf{n}}, \quad \forall m. \tag{22}$$

Thus, for the group of the plate vibration shapes a closed system of Eqs. (9), (16)–(18), (21), (22) is obtained. Write them with the phase indices $\alpha, \beta$ omitted:

$$\begin{aligned} &W_{mn} = I_{mn}(Q_{mn} - Q^{\delta s}_m - Q^{\delta r}_n), \quad I_{mn} = K^{-1}_{mn}, \\ &W^s_m = I^s_m Q^{\delta s}_m = \sum_n W_{mn}, \quad I^s_m = K^{s-1}_m, \\ &W^r_n = I^r_n Q^{\delta r}_n = \sum_m W_{mn}, \quad I^r_n = K^{r-1}_n. \end{aligned} \tag{23}$$

Here the compliances $I_{mn}$, $I^s_m$, $I^r_n$ are introduced for convenience. Eliminate the plate generalized displacements $W_{nm}$ from these equations by summing over $m$ and $n$ separately. As a result, the system of equations related to generalized forces $Q^{\delta s}_m$, $Q^{\delta r}_n$ acting on the stiffeners is obtained:

$$\begin{aligned} &\left(I^s_m + \sum_n I_{mn}\right) Q^{\delta s}_m + \sum_n I_{mn} Q^{\delta r}_n = \sum_n I_{mn} Q_{mn}, \\ &\left(I^r_n + \sum_m I_{mn}\right) Q^{\delta r}_n + \sum_m I_{mn} Q^{\delta s}_m = \sum_m I_{mn} Q_{mn}. \end{aligned} \tag{24}$$

The system can be rewritten in the matrix form:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}\begin{pmatrix} \mathbf{Q^S} \\ \mathbf{Q^R} \end{pmatrix} = \begin{pmatrix} \mathbf{W}_e^{\mathbf{S}} \\ \mathbf{W}_e^{\mathbf{R}} \end{pmatrix},$$

$$\mathbf{A}_{mm} = I_m^s + \sum_n I_{mn}, \quad \mathbf{B}_{mn} = I_{mn}, \quad \mathbf{Q}_m^{\mathbf{S}} = Q_m^{\delta s}, \quad \mathbf{W}_{em}^{\mathbf{S}} = \sum_n I_{mn} Q_{mn},$$

$$\mathbf{C}_{nm} = I_{mn}, \quad \mathbf{D}_{nn} = I_n^r + \sum_m I_{mn}, \quad \mathbf{Q}_n^{\mathbf{R}} = Q_n^{\delta r}, \quad \mathbf{W}_{en}^{\mathbf{R}} = \sum_m I_{mn} Q_{mn}. \tag{25}$$

Here **A**, **D** are the diagonal matrices; $\mathbf{C} = \mathbf{B}^{\mathrm{T}}$, T is the transpose. Matrix equation (25) effectively resolved as follows:

$$\mathbf{Q^R} = (\mathbf{D} - \mathbf{C}(\mathbf{A})^{-1}\mathbf{B})^{-1}(\mathbf{W}_e^{\mathbf{R}} - \mathbf{C}(\mathbf{A})^{-1}\mathbf{W}_e^{\mathbf{S}}),$$

$$\mathbf{Q^S} = (\mathbf{A})^{-1}(\mathbf{W}_e^{\mathbf{S}} - \mathbf{B}\mathbf{Q^R}). \tag{26}$$

Hence the stiffener responses $Q_m^{\delta s}$, $Q_n^{\delta r}$ are determined. Substituting them into the first expression in Eq. (23) resolves the task of predicting amplitudes $W_{mn}$ for one shape group of an infinite orthogonally stiffened plate. The total plate displacements $w(x,y)$ are obtained with integral (7).

In the case when the plate is strengthened by only one set of stiffeners, for example stringers, the independent groups will consist of the shapes with the same phase constant $\beta$ and the solution for them will be of the following form:

$$W_{nm} = I_{nm}(Q_{nm} - Q_m^{\delta s}), \quad Q_m^{\delta s} = \left(I_m^s + \sum_n I_{nm}\right)^{-1} \sum_n I_{nm} Q_{nm}, \quad (Q_n^{\delta r} \equiv 0). \tag{27}$$

In this case index $m$ means a continuous parameter.

In order to demonstrate the application of this method for determining the infinite stiffened plate vibrations, let us consider the plate exited by a single acoustic plane wave. Let the plane wave with frequency $\omega$ and amplitude $P$ be falling on the plate at some angle $\varphi$ between the plate plane and the wave vector and at angle $\theta$ between $x$-axis and the wave vector projection. The pressure acting on the plate presents the harmonic wave $p = A \exp \mathrm{i}\omega\{\cos(\varphi)(\cos(\theta)x + \sin(\theta)y)/c + t\}$. Here $c$ is the sound speed. The reaction of the medium from the sides of the plate is ignored for the sake of simplicity.

Here and in Section 4 the plate and rib parameters were chosen to be corresponding to the fuselage of a large passenger aircraft. The cell with dimensions of $d^r \times d^s = 0.5 \times 0.2m$ has the first eigen frequency 136 Hz. The loss tangent is taken to be $\eta = 0.03$ ($E = E_0(1+\mathrm{i}\eta)$).

Fig. 2 shows the mean-square velocity of an infinite stiffened plate (solid line 1) excited by a plane wave in a wide frequency range. The sound wave is impinging on the plate at angles $\varphi = \pi/8$, $\theta = \pi/4$. The velocity is normalized by velocity $V_m$ at purely inertial behavior of the unstiffened plate. For comparison the curves for an orthotropic plate with "smeared" ribs (dotted line) and for an isotropic unstiffened plate (dash line) are presented. The single maxima for isotropic or orthotropic plate models are explained by a coincidence of the wavenumber of the exiting field and the plate eigen-wavenumber. At low frequencies the stiffened plate behaves like the orthotropic one with smeared ribs. At high frequencies it behaves like the unstiffened one. The plate with discrete ribs manifests its highly resonance excitation at other frequencies. The interaction of a large number of vibration shapes is a reason of such behavior. The stiffened plate curve

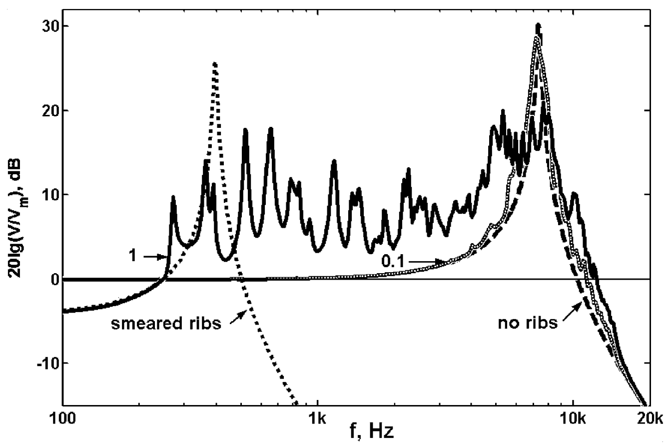


**Fig. 2.** Nondimensional mean-square velocity of the plate excited by a plane wave.

approaches to the curve for unstiffened one when the stiffeners become weaker. Solid line 0.1 is obtained for ribs with diminished cross-section sizes in 10 times.

## 3. Prediction relations for orthogonally stiffened cylindrical shell

Consider a part of the finite stiffened thin cylindrical shell of radius $R$, of length $L_x$ and of width $L_y$ (Fig. 3). Let it consist of $N^r$ spans with width $d^r$ between frames ($L_x = N^r d^r$). Each span consists of $N^s$ cells with width $d^s$ between stringers ($L_y = N^s d^s$). The stiffeners are connected with the shell along the lines. The shell is freely supported on the edges. The stiffeners split in half are left on the edges and take part in the deformation. This allows the finite shell to be considered as part of a shell, infinite in both directions.

The shell displacements $\mathbf{w}(x, y)$ are determined by three components and are related to the distributed forces acting on it through the equations of the shell vibrations, which can be written as follows:

$$\mathbf{Lw} = \mathbf{q} - \mathbf{q}^s - \mathbf{q}^r,$$

$$\mathbf{w}(x,y) = (u, v, w)^{\mathrm{T}}, \quad \mathbf{q}(x,y) = (q_u, q_v, q_w)^{\mathrm{T}}, \quad \mathbf{q}^{s,r}(x,y) = (q_u^{s,r}, q_v^{s,r}, q_w^{s,r})^{\mathrm{T}}, \tag{28}$$

where $\mathbf{L}$ is the elasto-inertial differential matrix operator of the shell with dimensions of $3 \times 3$, T is the transpose, $q(x, y)$, $q^s(x, y)$, $q^r(x, y)$ are the vectors of the external forces, stringer and frame response forces, respectively.

Vibrations of a separate stringer are determined by three displacement components and by the rotation around $x$-axis ($\theta^s$) and depend on the three components of forces ($q_u^{\delta s}, q_v^{\delta s}, q_w^{\delta s}$) and moment ($m_y^{\delta s}$), applied to the stringer:

$$\mathbf{L^s w^s} = \mathbf{q}^{\delta \mathbf{s}},$$

$$\mathbf{w^s}(x, p^s) = (u^s, v^s, w^s, \theta^s R)^T, \quad \mathbf{q}^{\delta \mathbf{s}}(x, p^s) = (q_u^{\delta s}, q_v^{\delta s}, q_w^{\delta s}, m_y^{\delta s}/R)^{\mathrm{T}}, \tag{29}$$

where $\mathbf{L^s}$ is the frequency-dependent elasto-inertial differential matrix operator of stringer with dimensions of $4 \times 4$. Owing to the insertion of radius $R$ in force and displacement vectors, the matrix operator $\mathbf{L^s}$ is symmetric. Vibrations of a separate frame can be written as follows:

$$\mathbf{L}^r \mathbf{w}^r = \mathbf{q}^{\delta r},$$

$$\mathbf{w}^r(p^r, y) = (u^r, v^r, w^r, \theta^r R)^T, \quad \mathbf{q}^{\delta r}(p^r, y) = (q_u^{\delta r}, q_v^{\delta r}, q_w^{\delta r}, m_x^{\delta r}/R)^{\mathrm{T}}, \tag{30}$$

where $\mathbf{L}^r$ is the frequency-dependent elasto-inertial differential matrix operator of frame.

The limited shell displacements and forces can be expanded in terms of special harmonic functions $\mathbf{\Phi_m}(x)\mathbf{\Psi_n}(y)$ (multiplier $\exp(i\omega t)$ is omitted everywhere, indices $\mathbf{mn}$ for vector components are also omitted here):

$$\mathbf{w}(x,y) = \sum_{\alpha,\beta} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mathbf{W_{mn}} \mathbf{\Phi_m}(x) \mathbf{\Psi_n}(y), \quad \mathbf{W_{mn}} = (U, V, W)^{\mathrm{T}}, \tag{31}$$

$$\mathbf{q} - \mathbf{q}^s - \mathbf{q}^r = \sum_{\alpha,\beta} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\mathbf{Q_{mn}} - \mathbf{Q^s_{mn}} - \mathbf{Q^r_{mn}}) \mathbf{\Phi_m}(x) \mathbf{\Psi_n}(y),$$

$$\mathbf{Q_{mn}} = (Q_U, Q_V, Q_W)^{\mathrm{T}}, \quad \mathbf{Q^{s,r}_{mn}} = (Q_U^{s,r}, Q_V^{s,r}, Q_W^{s,r})^{\mathrm{T}}, \tag{32}$$

$$\mathbf{\Phi_m}(x) = \begin{pmatrix} \cos(k_{\mathbf{m}} x) \\ \sin(k_{\mathbf{m}} x) \\ \sin(k_{\mathbf{m}} x) \end{pmatrix}, \quad \mathbf{\Psi_n}(y) = \begin{pmatrix} \sin(k_{\mathbf{n}} y) \\ -\cos(k_{\mathbf{n}} y) \\ \sin(k_{\mathbf{n}} y) \end{pmatrix}. \tag{33}$$

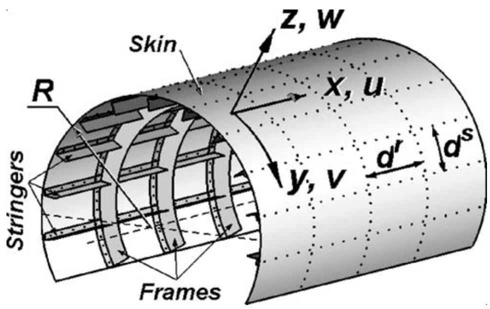


**Fig. 3.** Regularly stiffened finite cylindrical shell.

Here $\mathbf{W_{mn}}$ are the vectors of generalized displacements, $\mathbf{Q_{mn}}$, $\mathbf{Q}^s_{\mathbf{mn}}$, $\mathbf{Q}^r_{\mathbf{mn}}$ are the vectors of the generalized external forces, generalized stringer and frame response forces, respectively. In this paper the product of vectors $\mathbf{ab} \equiv (a_1b_1, a_2b_2, a_3b_3)^{\mathrm{T}}$ means a product terms by terms. The functions (or vibration shapes) in Eq. (33) have three components, satisfy the edge conditions and with properly defined component coefficients make up the unstiffened shell eigen-functions, that are not of interest here.

The wavenumbers $k_{\mathbf{m}}$, $k_{\mathbf{n}}$ are non-negative and multiples of $\pi/L_x$, $\pi/L_y$, respectively. The finite shell is considered as a part of an infinite shell. Any shape $\mathbf{\Phi_m\Psi_n}$ in Eq. (33) can be expressed as a sum of four exponential shapes with different signs of wavenumbers $\pm k_{\mathbf{m}}$, $\pm k_{\mathbf{n}}$. Hence, in contrast to the infinite-plate case, for the finite shell the independent shape groups are composed by all shapes, not only the wavenumber difference $(k_1{-}k_2)$, but also their sum $(k_1{+}k_2)$ is a multiple of $2\pi/d^r$ or $2\pi/d^s$.

The phase step $\alpha$ in the span between the frames can take only $N^r{+}1$ magnitudes and the phase step $\beta$ in the cell between the stringers takes only $N^s{+}1$ magnitudes:

$$\alpha = \{0, \pi/N^r, 2\pi/N^r, \ldots, \pi\},$$

$$\beta = \{0, \pi/N^s, 2\pi/N^s, \ldots, \pi\}. \tag{34}$$

This means that there exist $(N^r+1)(N^s+1)$ independent groups of shapes. Sort the wavenumbers in any group in ascending order:

$$k_{\mathbf{m}}d^r = \begin{cases} 2\pi(m-1) = \{0, 2\pi, 4\pi, \ldots\}, & \alpha = 0, \\ \pi\tilde{m} - (-1)^m\alpha = \{\alpha, 2\pi-\alpha, 2\pi+\alpha, \ldots\}, & 0<\alpha<\pi, \\ 2\pi(m-1)+\pi = \{\pi, 3\pi, 5\pi, \ldots\}, & \alpha = \pi, \end{cases}$$
$$m = 1, 2, \ldots, \quad \tilde{m} = m + ((-1)^m - 1)/2 = \{0, 2, 2, 4, 4, \ldots\},$$

$$k_{\mathbf{n}}d^s = \begin{cases} 2\pi(n-1), & \beta = 0, \\ \pi\tilde{n} - (-1)^n\beta, & 0<\beta<\pi, \\ 2\pi(n-1)+\pi, & \beta = \pi, \end{cases} \quad n = 1, 2, \ldots, \quad \tilde{n} = n + ((-1)^n - 1)/2. \tag{35}$$

Using expansions (31) and (32) and the equation for the shell (28), we get a relation between the generalized displacement vectors and forces acting on the shell:

$$\mathbf{L}(\omega)\mathbf{\Phi_m}(x)\mathbf{\Psi_n}(y) = \mathbf{K_{mn}}(\omega)\mathbf{\Phi_m}(x)\mathbf{\Psi_n}(y), \tag{36}$$

$$\mathbf{W_{mn}} = \mathbf{I_{mn}}(\mathbf{Q_{mn}} - \mathbf{Q}^s_{\mathbf{mn}} - \mathbf{Q}^r_{\mathbf{mn}}), \quad \mathbf{I_{mn}} = (\mathbf{K_{mn}})^{-1}, \tag{37}$$

where $\mathbf{K_{mn}}$ is the matrix of frequency-dependent coefficients with dimensions of $3\times3$. In Appendix A the expressions for matrices of coefficients $\mathbf{K_{mn}}$, that were used in calculations, are presented, see Eq. (A.1). $\mathbf{I_{mn}}$ is the compliancy matrix inverse to $\mathbf{K_{mn}}$.

Expand the displacements of all the stringers and the forces acting on them in terms of the following functions satisfying the edge conditions:

$$\mathbf{w}^s(x, p^s) = \sum_{\alpha,\beta}\sum_{m=1}^{\infty} \mathbf{W}^s_{\beta\mathbf{m}}\mathbf{\Phi}^s_{\mathbf{m}}(x)\mathbf{\Psi}^s_{\beta}(p^s), \tag{38}$$

$$\mathbf{q}^{\delta s}(x, p^s) = d^s\sum_{\alpha,\beta}\sum_{m=1}^{\infty} \mathbf{Q}^{\delta s}_{\beta\mathbf{m}}\mathbf{\Phi}^s_{\mathbf{m}}(x)\mathbf{\Psi}^s_{\beta}(p^s), \tag{39}$$

$$\mathbf{W}^s_{\beta\mathbf{m}} = \begin{pmatrix} U^s \\ V^s \\ W^s \\ \Theta^s R \end{pmatrix}, \quad \mathbf{Q}^{\delta s}_{\beta\mathbf{m}} = \begin{pmatrix} Q^s_U \\ Q^s_V \\ Q^s_W \\ M^s R^{-1} \end{pmatrix}, \quad \mathbf{\Phi}^s_{\mathbf{m}}(x) = \begin{pmatrix} \cos(k_{\mathbf{m}}x) \\ \sin(k_{\mathbf{m}}x) \\ \sin(k_{\mathbf{m}}x) \\ \sin(k_{\mathbf{m}}x) \end{pmatrix}, \quad \mathbf{\Psi}^s_{\beta}(p^s) = \begin{pmatrix} \sin(p^s\beta) \\ -\cos(p^s\beta) \\ \sin(p^s\beta) \\ \cos(p^s\beta) \end{pmatrix}. \tag{40}$$

The first three components of function $\mathbf{\Phi}^s_{\mathbf{m}}$ coincide with those of shell function $\mathbf{\Phi}^s_{\mathbf{m}}(x) = (\mathbf{\Phi_m}(x)^{\mathrm{T}}, \sin(k_{\mathbf{m}}x))^{\mathrm{T}}$. The discrete function $\mathbf{\Psi}^s_{\beta}(p^s)$ serves for correlating the vibrations of all the stringers with the shell vibrations in a group. Its first three components coincide with those of the first in this group of shell functions on the stringer locations $\mathbf{\Psi}^s_{\beta}(p^s) = (\mathbf{\Psi}_{\beta,n=1}(p^sd^s)^{\mathrm{T}}, \cos(p^s\beta))^{\mathrm{T}}$. In contrast to the shell in the case of considering the stringer, the summation in the group is made only over index $m$. Using expansions (38)–(40) and Eq. (29), we get a connection between the generalized vectors of displacements and forces acting on the stringers:

$$\mathbf{L}^s(\omega)\mathbf{\Phi}^s_{\mathbf{m}}(x) = d^s\mathbf{K}^s_{\mathbf{m}}(\omega)\mathbf{\Phi}^s_{\mathbf{m}}(x), \tag{41}$$

$$\mathbf{W}^s_{\beta\mathbf{m}} = \mathbf{I}^s_{\mathbf{m}}\mathbf{Q}^{\delta s}_{\beta\mathbf{m}}, \quad \mathbf{I}^s_{\mathbf{m}} = (\mathbf{K}^s_{\mathbf{m}})^{-1}. \tag{42}$$

Here $\mathbf{K}^s_{\mathbf{m}}$ is the matrix of frequency-dependent coefficients with dimensions of $4\times4$ for stringer, see Eq. (A.3). $\mathbf{I}^s_{\mathbf{m}}$ is the compliancy matrix inverse to $\mathbf{K}^s_{\mathbf{m}}$.

Present the vibrations of all the frames in the form of the following expansion:

$$\mathbf{w}^r(p^r,y) = \sum_{\alpha\beta}\sum_{n=1}^{\infty}\mathbf{W}^r_{\alpha\mathbf{n}}\mathbf{\Phi}^r_{\alpha}(p^r)\mathbf{\Psi}^r_{\mathbf{n}}(y), \tag{43}$$

$$\mathbf{q}^{\delta r}(p^r,y) = d^r\sum_{\alpha\beta}\sum_{n=1}^{\infty}\mathbf{Q}^{\delta r}_{\alpha\mathbf{n}}\mathbf{\Phi}^r_{\alpha}(p^r)\mathbf{\Psi}^r_{\mathbf{n}}(y), \tag{44}$$

$$\mathbf{\Phi}^r_{\alpha}(p^r) = \begin{pmatrix}\cos(p^r\alpha)\\ \sin(p^r\alpha)\\ \sin(p^r\alpha)\\ \cos(p^r\alpha)\end{pmatrix}, \quad \mathbf{\Psi}^r_{\mathbf{n}}(y) = \begin{pmatrix}\sin(k_{\mathbf{n}}y)\\ -\cos(k_{\mathbf{n}}y)\\ \sin(k_{\mathbf{n}}y)\\ \sin(k_{\mathbf{n}}y)\end{pmatrix}. \tag{45}$$

Note here, that $\mathbf{\Phi}^r_{\alpha}(p^r) = (\mathbf{\Phi}_{\alpha,m=1}(p^rd^r)^{\mathrm{T}}, \cos(p^r\alpha))^{\mathrm{T}}$, $\mathbf{\Psi}^r_{\mathbf{n}}(y) = (\mathbf{\Psi}_{\mathbf{n}}(y)^{\mathrm{T}}, \sin(k_{\mathbf{n}}y))^{\mathrm{T}}$. Substitution of expansion (43) in frame-vibration equation (30) leads to the relation between the generalized vectors of frame displacements and forces:

$$\mathbf{L}^r(\omega)\mathbf{\Psi}^r_{\mathbf{n}}(y) = d^r\mathbf{K}^r_{\mathbf{n}}(\omega)\mathbf{\Psi}^r_{\mathbf{n}}(y), \tag{46}$$

$$\mathbf{W}^r_{\alpha\mathbf{n}} = \mathbf{I}^r_{\mathbf{n}}\mathbf{Q}^{\delta r}_{\alpha\mathbf{n}}, \quad \mathbf{I}^r_{\mathbf{n}} = (\mathbf{K}^r_{\mathbf{n}})^{-1}. \tag{47}$$

Here $\mathbf{K}^r_{\mathbf{n}}$ is the matrix of coefficients for frame with dimensions of $4\times4$, see Eq. (A.4). $\mathbf{I}^r_{\mathbf{n}}$ is the matrix inverse to $\mathbf{K}^r_{\mathbf{n}}$.

We have considered the vibrations of the shell, the system of stringers and the system of frames separately. Now, in order to obtain a unified system of equations, the displacements of the shell, the stringers and the frames must be connected. The generalized stringer and frame displacements $\mathbf{W}^s_{\beta\mathbf{m}}, \mathbf{W}^r_{\alpha\mathbf{n}}$ can be expressed through the shell displacements $\mathbf{W}_{\mathbf{mn}}$ with the help of special matrices $\mathbf{E}_{\mathbf{n}}, \mathbf{E}_{\mathbf{m}}$ of dimension $4\times3$, that are obtained in Appendix B, see Eqs. (B.5) and (B.7):

$$\mathbf{W}^s_{\beta\mathbf{m}} = \sum_n \mathbf{E}_{\mathbf{n}}\mathbf{W}_{\mathbf{mn}}, \quad \mathbf{W}^r_{\alpha\mathbf{n}} = \sum_m \mathbf{E}_{\mathbf{m}}\mathbf{W}_{\mathbf{mn}}. \tag{48}$$

The relation between the vector of generalized stringer responses $\mathbf{Q}^s_{\mathbf{mn}}$ and the forces $\mathbf{Q}^{\delta s}_{\beta\mathbf{m}}$ can be expressed through the special $\mathbf{F}_{\mathbf{n}}$ matrix, that is also obtained in Appendix B. The relation between generalized frames responses $\mathbf{Q}^r_{\mathbf{mn}}$ and the forces $\mathbf{Q}^{\delta r}_{\alpha\mathbf{n}}$—through $\mathbf{F}_{\mathbf{m}}$ matrix, see Eqs. (B.14) and (B.15):

$$\mathbf{Q}^s_{\mathbf{mn}} = \mathbf{F}_{\mathbf{n}}\mathbf{Q}^{\delta s}_{\beta\mathbf{m}}, \quad \mathbf{Q}^r_{\mathbf{mn}} = \mathbf{F}_{\mathbf{m}}\mathbf{Q}^{\delta r}_{\alpha\mathbf{n}}. \tag{49}$$

Thus, we have obtained the closed system of Eqs. (37), (42), (47)–(49). They are rewritten, omitting the phase indices $\alpha$, $\beta$ excluding the generalized displacements $\mathbf{W}^s_m, \mathbf{W}^r_n$ and force responses $\mathbf{Q}^s_{mn}, \mathbf{Q}^r_{mn}$ of the stiffeners in the following form:

$$\mathbf{W}_{mn} = \mathbf{I}_{mn}(\mathbf{Q}_{mn} - \mathbf{F}_n\mathbf{Q}^{\delta s}_m - \mathbf{F}_m\mathbf{Q}^{\delta r}_n),$$

$$\mathbf{I}^r_n\mathbf{Q}^{\delta r}_n = \sum_m \mathbf{E}_m\mathbf{W}_{mn}, \quad \mathbf{I}^s_m\mathbf{Q}^{\delta s}_m = \sum_n \mathbf{E}_n\mathbf{W}_{mn}. \tag{50}$$

Substituting the first equality in Eq. (50) into second and third we get a system of equations related to the generalized forces acting on the stiffeners, which is more conveniently recorded in a matrix form:

$$\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{pmatrix}\begin{pmatrix}\mathbf{Q}^{\mathbf{S}}\\ \mathbf{Q}^{\mathbf{R}}\end{pmatrix} = \begin{pmatrix}\mathbf{W}^{\mathbf{S}}_{\mathbf{e}}\\ \mathbf{W}^{\mathbf{R}}_{\mathbf{e}}\end{pmatrix},$$

$$\mathbf{A}_{m'm'} = \mathbf{I}^s_m + \sum_n \mathbf{E}_n\mathbf{I}_{mn}\mathbf{F}_n, \quad \mathbf{B}_{m'n'} = \mathbf{E}_n\mathbf{I}_{mn}\mathbf{F}_m, \quad \mathbf{Q}^{\mathbf{S}}_{m'} = \mathbf{Q}^{\delta s}_m,$$

$$\mathbf{C}_{n'm'} = \mathbf{E}_m\mathbf{I}_{mn}\mathbf{F}_n, \quad \mathbf{D}_{n'n'} = \mathbf{I}^r_n + \sum_m \mathbf{E}_m\mathbf{I}_{mn}\mathbf{F}_m, \quad \mathbf{Q}^{\mathbf{R}}_{n'} = \mathbf{Q}^{\delta r,\alpha\beta}_n,$$

$$\mathbf{W}^{\mathbf{S}}_{\mathbf{e}m'} = \sum_n \mathbf{E}_n\mathbf{I}_{mn}\mathbf{Q}_{mn}, \quad \mathbf{W}^{\mathbf{R}}_{\mathbf{e}n'} = \sum_m \mathbf{E}_m\mathbf{I}_{mn}\mathbf{Q}_{mn}, \quad m' = 4(m-1)+\{1,2,3,4\}, \; n' = 4(n-1)+\{1,2,3,4\}. \tag{51}$$

Here **A**, **D** are matrices with $4 \times 4$-dimensional blocks on the diagonal, matrices **B**, **C** consist of $4 \times 4$-dimensional blocks. Vectors $\mathbf{Q^S}$, $\mathbf{Q^R}$, $\mathbf{W_e^S}$, $\mathbf{W_e^R}$ consist of four-dimensional sub-vectors. Vectors $\mathbf{W_e^S}$, $\mathbf{W_e^R}$ imply the generalized displacements of absolutely compliant ribs on the plate. Formally the system of Eqs. (51) differs from the system (25) for the infinite stiffened plate only by additional matrix multipliers $\mathbf{E}_m$, $\mathbf{E}_n$, $\mathbf{F}_m$, $\mathbf{F}_n$.

We restrict ourselves to a certain number of shapes in group ($n \le n_{\max}, m \le m_{\max}$). Then the system of Eqs. (51) can be resolved as

$$\mathbf{Q^R} = (\mathbf{D} - \mathbf{C}(\mathbf{A})^{-1}\mathbf{B})^{-1}(\mathbf{W_e^R} - \mathbf{C}(\mathbf{A})^{-1}\mathbf{W_e^S}),$$

$$\mathbf{Q^S} = (\mathbf{A})^{-1}(\mathbf{W_e^S} - \mathbf{B}\mathbf{Q^R}). \tag{52}$$

Substitution of the obtained stiffener responses into Eq. (50) gives the sought for vectors of generalized shell displacements $\mathbf{W}_{mn}$.

From expression (51), as a special case, the solution for the shell stiffened by only one set of stiffeners, for example stringers, can be directly written down. The generalized shell displacements for one shape group with some phase constant $\beta$ are determined in this case as follows:

$$\mathbf{W}_{mn} = \mathbf{I}_{mn}(\mathbf{Q}_{mn} - \mathbf{F}_n\mathbf{Q}_m^{\delta s}) \quad \text{(only stringers)},$$

$$\mathbf{Q}_m^{\delta s} = \left(\mathbf{I}_m^s + \sum_n \mathbf{E}_n\mathbf{I}_{mn}\mathbf{F}_n\right)^{-1} \sum_n \mathbf{E}_n\mathbf{I}_{mn}\mathbf{Q}_{mn}. \tag{53}$$

Here $k_m = \pi m/L_x$ in matrices $\mathbf{I}_{mn}$, $\mathbf{I}_m^s$.

Thus, the task related to vibrations of the orthogonally stiffened shell with accounting for the interconnection between all the components of displacements and forces of the shell and of the regularly spaced stiffeners is solved. Note, that the external force vectors $\mathbf{q} = (q_u, q_v, q_w)^{\mathrm{T}}$ have three components and hence one can predict the shell vibrations caused not only by the normal force fields, but also caused by fields of forces with arbitrary direction.

An additional benefit of the proposed method based on special harmonic expansions is the fact that the result of vibration predictions is presented in the form of amplitudes of sinusoidal shapes. This makes it possible to proceed directly to solving the tasks related to sound wave radiation or to internal acoustic mode excitation, using the methods of predicting the internal noise, worked out earlier [7,8].

## 4. Examples of prediction of stiffened shell vibrations

Now we demonstrate an application of the vibration–calculation method using as an example a stiffened shell excited by a point force. The point force produces a uniform picture of generalized forces in the whole wavenumber space. Therefore such an action is convenient for testing the prediction program and determining the main properties of excitation of the structure.

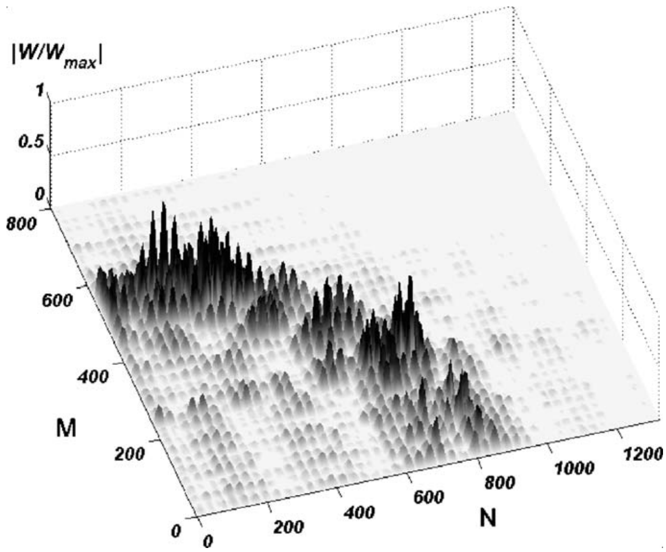


**Fig. 4.** An example of predicting the generalized displacements $W_{nm}$ of nearly million shapes under point excitation at frequency of $f = 20$ kHz.

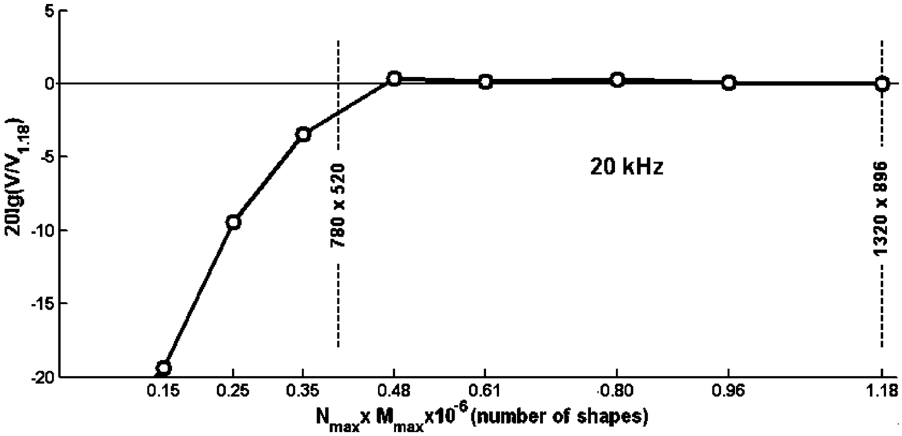


**Fig. 5.** Convergence of the prediction results with a growth of shape number, $f = 20\,\text{kHz}$.

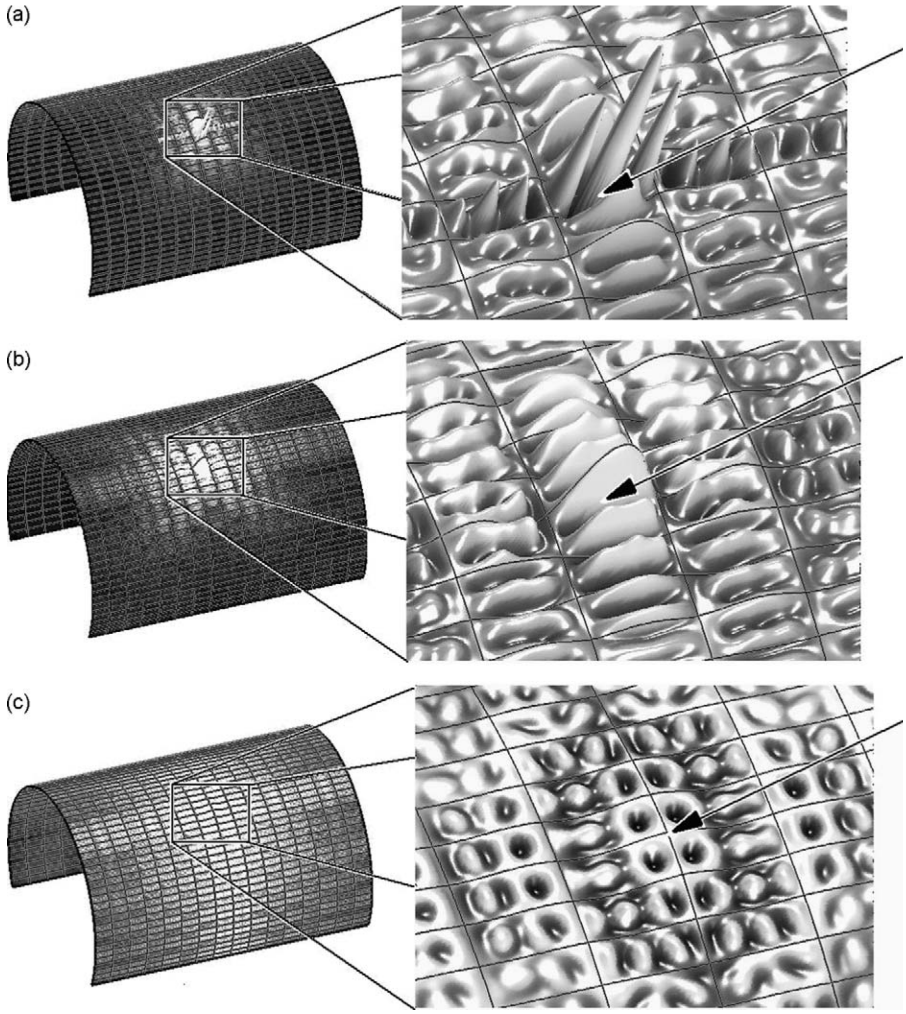


**Fig. 6.** Excitation of the stiffened shell at frequency $f = 1000\,\text{Hz}$ by a harmonic point force applied to the: (a) cell center, (b) stringer center and (c) stiffener intersection.

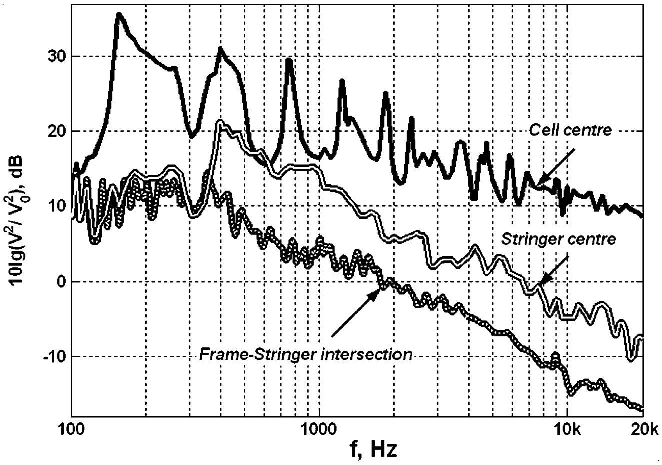


**Fig. 7.** Mean velocity of the shell excited by a point force $q_0\,e^{i\omega t}$ applied to three different positions. $V_0 = 2.24\times10^{-3}$ m/s if $q_0 = 1$ N.

To make the calculations, the structural parameters were chosen which correspond to a part of the fuselage of a large passenger aircraft. Its radius is $R = 3$ m, length $L_x = 8$ m, corresponding to $N^s = 16$ spans with length $d^r = 0.5$ m and its width is $L_y = 12$ m corresponding to $N^r = 60$ cells in each span with $d^s = 0.2$ m. The loss tangent is taken to be $\eta = 0.1$ for Figs. 4 and 6 and $\eta = 0.03$ for Figs. 5 and 7.

The number of shapes required for prediction was determined directly from the result of illustrative predictions of displacement amplitudes. Fig. 4 presents an example of such a test prediction for the maximum of considered frequencies (20 kHz) under point force excitation. A point force with coordinates $x_0 = 3.75$ m, $y_0 = 4.3$ m was applied to the cell center. A quarter of the ellipse produced by the dominant shapes in Fig. 4 corresponds to the wavenumbers being determined from the relation for traveling waves in the plate $(k_x^2 + k_y^2)^2 = m\omega^2/D$. The prediction accounted for $N_{max}\times M_{max} = 22N^r\times 50N^s = 1320\times 800 = 1\,056\,000$ of vibration shapes. The principal difficulty of the calculations relates to the $(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})$ matrix inversion. In this case all 1 056 000 shapes are broken into $(N^r{+}1)\times(N^s{+}1) = 17\times 61$ groups, each of them consisting maximum of $22\times 50$ shapes. Matrix $(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})$ has dimensions $88\times 88$ $(88 = 4\times 22)$.

Another, more commonly used method of determining the sufficient number of shapes, consists in the investigation of convergence of the prediction results with a growth of shape number. Fig. 5 illustrates the convergence of mean-square velocity value for such a point force excitation. The ratio of the maximum number of shapes in two directions was determined by the relation $N_{max}/M_{max}\approx L_y/L_x$. We can see, that the maximum indices of shapes must substantially exceed the indices of dominating shapes $\pi N_{max}/L_x \gg \sqrt{[4]}m\omega^2/D$, $\pi M_{max}/L_y \gg \sqrt{[4]}m\omega^2/D$ ($N_{max}\gg 780$, $M_{max}\gg 520$ in this case for 20 kHz) in order to get the reliable results.

Fig. 6a–c give an example of a snap-shot displacement prediction for such a piece of structure at frequency of 1000 Hz, produced by a point force applied to the cell center, the stringer middle and a stiffener intersection. All three figures are made with the same displacement scale. In the first case (Fig. 6a) an intensive excitation of the central and the neighboring cells occurs. In the second case, when the force is applied to a stringer (Fig. 6b), an intensive wave propagation over the stringers is observed. In the third case, when a force is applied to the intersection of a stringer and a frame (Fig. 6c), an intensive wave propagation in all directions is observed, though the displacement amplitude is small.

Fig. 7 shows the frequency dependence of the rms-velocity of the shell vibrations in the frequency range 100–20 000 Hz for the above three cases of excitation by a point force. The largest vibration velocity over the whole frequency range is caused by the force applied to the cell center. The maximum is achieved at the frequency slightly greater than the frequency of an isolated simply supported cell (136 Hz). The smallest vibrations are caused by the force applied to the stiffener intersection. It should be noted that at the frequencies less than the frequency of an isolated stringer span (440 Hz), the force applied to the intersection and to the middle of a stringer causes practically identical rms-velocities of the shell.

## 4. Conclusion

The task related to forced vibrations of the cylindrical shell with an orthogonal system of stiffeners is solved with a correct account for their discreteness and elasto-inertial properties. The solution which is compact and permits determining all the components of construction vibration velocities directly under excitation by normal and tangential forces is obtained. These components are presented in the form of special double trigonometric series. Such a presentation of velocities substantially simplifies the solution of subsequent tasks related to acoustic radiation of panels and shells and to forming the acoustic field inside a closed volume. Illustrative examples of the vibrations of framed shell modeling a large fragment of the aircraft fuselage section at point force excitation demonstrate a high efficiency of applying this method over the whole sound frequency range.

## Appendix A. Matrices of coefficients

The elasto-inertial operators of the shell and the stiffeners with rectangular cross-section according to Ref. [9] were used. In all the expressions the following values are used: $m$, $h$, $R$, $E$, $\mu$ are the shell surface mass, thickness, radius, Young's modulus, Poisson's ratio, respectively; $\omega_{\text{Ring}} = \sqrt{K/mR^2}$ is the shell ring frequency; $K = Eh/(1-\mu^2)$ is the shell rigidity in tension; $D = Eh^3/12(1-\mu^2)$ is the cylindrical rigidity; $a^2 = h^2/(12R^2)$ is the thickness coefficient. All the matrices of coefficients are expressed through the dimensionless values. The following dimensionless values are also common for all the equations here: $\bar{m} = Rk_{\mathbf{m}}$, $\bar{n} = Rk_{\mathbf{n}}$ are the dimensionless longitudinal and circumferential wavenumbers; $\bar{\omega} = \omega/\omega_{\text{Ring}}$ is the dimensionless circular frequency. For the shell vibration shapes (Eq. (33)), the matrix of coefficients $\mathbf{K_{mn}}$ is as follows:

$$\mathbf{K_{mn}} = m\omega^2_{\text{Ring}}\left(\begin{pmatrix} \bar{m}^2+\mu^-\bar{n}^2 & -\mu^+\bar{m}\bar{n} & -\mu\bar{m} \\ -\mu^+\bar{m}\bar{n} & \bar{K}_{22} & \bar{K}_{23} \\ -\mu\bar{m} & \bar{K}_{23} & \bar{K}_{33} \end{pmatrix} - \bar{\omega}^2\right),$$

$$\bar{K}_{22} = \mu^-\bar{m}^2 + \bar{n}^2 + a^2(4\mu^-\bar{m}^2+\bar{n}^2), \quad \mu^\pm = (1\pm\mu)/2,$$

$$\bar{K}_{23} = \bar{n} + a^2\bar{n}((2-\mu)\bar{m}^2+\bar{n}^2), \quad \bar{K}_{33} = a^2(\bar{m}^2+\bar{n}^2)^2+1. \tag{A.1}$$

The stringers and frames are characterized by: Young's moduli $E^s$, $E^r$; Poisson's ratios $\mu^s$, $\mu^r$; masses per unit length $m^s$, $m^r$; eccentricities $z^s$, $z^r$ positive for the internal position; $A^{s,r}$, $I^{s,r}$, $I_Z^{s,r}$, $J^{s,r}$ are the areas, the principal moments, moments relative to $z$-axis and the polar moments for cross-sections, respectively; $G^{s,r} = E^{s,r}/(2(1+\mu^{s,r}))$ is the shear rigidity. The following dimensionless values are used later:

$$\bar{m}^{s,r} \equiv \frac{m^{s,r}}{md^{s,r}}, \quad \bar{z}^{s,r} \equiv \frac{z^{s,r}}{R}, \quad \bar{r}^{s,r} \equiv 1-\bar{z}^{s,r},$$

$$\bar{E}^{s,r} \equiv \frac{E^{s,r}A^{s,r}}{d^{s,r}K}, \quad \bar{G}^{s,r} \equiv \frac{G^{s,r}J^{s,r}}{d^{s,r}D}, \quad \bar{D}^{s,r} \equiv \frac{E^{s,r}I^{s,r}}{d^{s,r}D},$$

$$D_Z^{s,r} \equiv \frac{E^{s,r}I_Z^{s,r}}{d^{s,r}D}, \quad \bar{J}^{s,r} \equiv \frac{J^{s,r}}{A^{s,r}R^2}. \tag{A.2}$$

For the stringer vibration shape (Eq. (40)) the matrix of coefficients is as follows:

$$\mathbf{K^s_m} = m\omega^2_{\text{Ring}}(\bar{\mathbf{K}}^s - \bar{m}^s\bar{\omega}^2\bar{\mathbf{M}}^s),$$

$$\bar{\mathbf{K}}^s = \begin{pmatrix} \bar{E}^s\bar{m}^2 & 0 & \bar{z}^s\bar{E}^s\bar{m}^3 & 0 \\ 0 & \bar{K}^s_{22} & 0 & \bar{K}^s_{24} \\ \bar{z}^s\bar{E}^s\bar{m}^3 & 0 & \bar{K}^s_{33} & 0 \\ 0 & \bar{K}^s_{24} & 0 & \bar{K}^s_{44} \end{pmatrix}, \quad \bar{\mathbf{M}}^s = \begin{pmatrix} 1 & 0 & \bar{z}^s\bar{m} & 0 \\ 0 & \bar{M}^s_{22} & 0 & \bar{M}^s_{24} \\ \bar{z}^s\bar{m} & 0 & \bar{M}^s_{33} & 0 \\ 0 & \bar{M}^s_{24} & 0 & \bar{M}^s_{44} \end{pmatrix},$$

$$\bar{K}^s_{22} = a^2\bar{m}^2(\bar{G}^s + \bar{r}^{s2}\bar{D}^s_Z\bar{m}^2), \quad \bar{M}^s_{22} = \bar{r}^s + \bar{J}^s,$$

$$\bar{K}^s_{24} = a^2\bar{m}^2(-\bar{G}^s + \bar{r}^s\bar{D}^s_Z\bar{z}^s\bar{m}^2), \quad \bar{M}^s_{24} = \bar{z}^s\bar{r}^s - \bar{J}^s,$$

$$\bar{K}^s_{33} = \bar{m}^4(a^2\bar{D}^s + \bar{z}^{s2}\bar{E}^s), \quad \bar{M}^s_{33} = 1 + \bar{z}^{s2}\bar{m}^2$$

$$\bar{K}^s_{44} = a^2\bar{m}^2(\bar{G}^s + \bar{D}^s_Z\bar{z}^{s2}\bar{m}^2), \quad \bar{M}^s_{44} = \bar{z}^{s2} + \bar{J}^s. \tag{A.3}$$

For the frame vibration shape (Eq. (45)) the matrix of coefficients is the following:

$$\mathbf{K^r_n} = m\omega^2_{\text{Ring}}(\bar{\mathbf{K}}^r - \bar{m}_r\bar{\omega}^2\bar{\mathbf{M}}^r),$$

$$\bar{\mathbf{K}}^r = \begin{pmatrix} \bar{K}^r_{11} & 0 & 0 & \bar{K}^r_{14} \\ 0 & \bar{K}^r_{22} & \bar{K}^r_{23} & 0 \\ 0 & \bar{K}^r_{23} & \bar{K}^r_{33} & 0 \\ \bar{K}^r_{14} & 0 & 0 & \bar{K}^r_{44} \end{pmatrix}, \quad \bar{\mathbf{M}}^r = \begin{pmatrix} 1 & 0 & 0 & \bar{z} \\ 0 & \bar{r}^r & -\bar{z}_r\bar{r}^r\bar{n} & 0 \\ 0 & -\bar{z}_r\bar{r}^r\bar{n} & 1+\bar{z}_r^2\bar{n}^2 & 0 \\ \bar{z} & 0 & 0 & \bar{z}_r^2+\bar{J}_r \end{pmatrix},$$

$$\bar{K}_{11}^r = a^2\bar{n}^2(\bar{G}^r + \bar{D}_Z^r\bar{n}^2), \quad \bar{K}_{14}^r = a^2\bar{n}^2(\bar{D}_Z^r(1+\bar{z}^r\bar{n}^2) + \bar{G}^r(1+\bar{z}^r)),$$

$$\bar{K}_{22}^r = \bar{r}^{r2}\bar{E}^r\bar{n}^2, \quad \bar{K}_{23}^r = \bar{r}^r\bar{E}^r\bar{n}(1-\bar{z}^r\bar{n}^2),$$

$$\bar{K}_{33}^r = a^2\bar{D}^r(1-\bar{n}^2)^2 + \bar{E}^r(1-\bar{z}^r\bar{n}^2)^2, \quad \bar{K}_{44}^r = a^2((1+\bar{z}^r)^2\bar{G}^r\bar{n}^2 + \bar{D}_Z^r(1+\bar{z}^r\bar{n}^2)^2). \tag{A.4}$$

## Appendix B. Connection of displacements and forces

The relation of the displacements and rotation of the stiffeners and of those of the shell along the stiffness is as follows:

$$\mathbf{w}^s(x,p^s) \equiv (u^s, v^s, w^s, \theta^s R)^{\mathrm{T}} = (u, v, w, w'_y R)^{\mathrm{T}} \equiv \mathbf{w}(x, p^s d^s), \tag{B.1}$$

$$\mathbf{w}^r(p^r,y) \equiv (u^r, v^r, w^r, \theta^r R)^{\mathrm{T}} = (u, v, w, w'_x R)^{\mathrm{T}} \equiv \mathbf{w}(p^r d^r, y). \tag{B.2}$$

Here the stiffener rotation angles $\theta^s$, $\theta^r$ are equal to the derivatives of the shell normal displacements in the respective coordinates $(w'_x, w'_y)$. The stringer displacements and rotation can be expressed through the shell generalized displacements ($U$, $V$, $W$) as follows:

$$\mathbf{w}^s(x,p^s) = \sum_{\alpha,\beta}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\begin{pmatrix} U_{\mathbf{mn}} \\ V_{\mathbf{mn}} \\ W_{\mathbf{mn}} \\ k_{\mathbf{n}}RW_{\mathbf{mn}} \end{pmatrix}\begin{pmatrix} \cos(k_{\mathbf{m}}x) \\ \sin(k_{\mathbf{m}}x) \\ \sin(k_{\mathbf{m}}x) \\ \sin(k_{\mathbf{m}}x) \end{pmatrix}\begin{pmatrix} \sin(k_{\mathbf{n}}p^s d^s) \\ -\cos(k_{\mathbf{n}}p^s d^s) \\ \sin(k_{\mathbf{n}}p^s d^s) \\ \cos(k_{\mathbf{n}}p^s d^s) \end{pmatrix}. \tag{B.3}$$

Taking into account that

$$\begin{aligned}\cos(k_{\mathbf{n}}p^s d^s) &= \cos(p^s\beta), \\ \sin(k_{\mathbf{n}}p^s d^s) &= s_{\mathbf{n}}\,\sin(p^s\beta),\end{aligned} \qquad s_{\mathbf{n}} = \begin{cases} (-1)^{n+1}, & \beta \neq 0, \pi, \\ 0, & \beta = 0, \pi, \end{cases} \tag{B.4}$$

and comparing Eq. (B.3) with expansion (38) for the stringers, the generalized stringer displacements $\mathbf{W}^s_{\beta\mathbf{m}}$ can be expressed through the shell displacements $\mathbf{W}_{\mathbf{mn}}$ as

$$\mathbf{W}^s_{\beta\mathbf{m}} = \sum_n \begin{pmatrix} s_{\mathbf{n}}U_{\mathbf{mn}} \\ V_{\mathbf{mn}} \\ s_{\mathbf{n}}W_{\mathbf{mn}} \\ k_{\mathbf{n}}RW_{\mathbf{mn}} \end{pmatrix} = \sum_n \begin{pmatrix} s_{\mathbf{n}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s_{\mathbf{n}} \\ 0 & 0 & k_{\mathbf{n}}R \end{pmatrix}\begin{pmatrix} U_{\mathbf{mn}} \\ V_{\mathbf{mn}} \\ W_{\mathbf{mn}} \end{pmatrix} \equiv \sum_n \mathbf{E_n W_{mn}}. \tag{B.5}$$

Here $\mathbf{E_n}$ is the matrix, of dimension $4\times 3$, relating the vector of generalized displacements of the shell and the stringers. Similarly the relation between generalized frame and shell displacements can be obtained:

$$\mathbf{W}^r_{\alpha\mathbf{n}} = \sum_m \mathbf{E_m W_{mn}}, \tag{B.6}$$

$$\mathbf{E_m} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & s_{\mathbf{m}} & 0 \\ 0 & 0 & s_{\mathbf{m}} \\ 0 & 0 & k_{\mathbf{m}}R \end{pmatrix}, \quad s_{\mathbf{m}} = \begin{cases} (-1)^{m+1}, & \alpha \neq 0, \pi, \\ 0, & \alpha = 0, \pi. \end{cases} \tag{B.7}$$

Here $\mathbf{E_m}$ indicates the matrix, of dimensions $4\times 3$, relating the vector of generalized displacements of the shell and the frame.

Now we find a relation of the surface forces of the stiffener responses and the discrete forces acting on the stiffeners. These discrete forces can be transformed into the surface forces in a similar way to the plate case. Account for the equality similar to Eq. (19):

$$d^r \sum_{p^r}\begin{pmatrix} \sin(\alpha p^r) \\ \cos(\alpha p^r) \end{pmatrix}\delta(x - p^r d^r) = \sum_{m=1}^{\infty}\begin{pmatrix} s_{\mathbf{m}}\,\sin(k_{\mathbf{m}}x) \\ \varepsilon_{\mathbf{m}}\,\cos(k_{\mathbf{m}}x) \end{pmatrix}, \tag{B.8}$$

$$d^s \sum_{p^s}\begin{pmatrix} \sin(\beta p^s) \\ \cos(\beta p^s) \end{pmatrix}\delta(y - p^s d^s) = \sum_{m=1}^{\infty}\begin{pmatrix} s_{\mathbf{n}}\,\sin(k_{\mathbf{n}}y) \\ \varepsilon_{\mathbf{n}}\,\cos(k_{\mathbf{n}}y) \end{pmatrix}, \tag{B.9}$$

$$\varepsilon_{\mathbf{m}} = \begin{cases} 1, & 0<\alpha<\pi;\ \alpha = 0, m = 1, \\ 2, & \alpha = \pi;\ \alpha = 0, m>1, \end{cases} \quad \varepsilon_{\mathbf{n}} = \begin{cases} 1, & 0<\beta<\pi;\ \beta = 0, n = 1, \\ 2, & \beta = \pi;\ \beta = 0, n>1. \end{cases} \tag{B.10}$$

Note here, that $\varepsilon_{\mathbf{m}}s_{\mathbf{m}} = s_{\mathbf{m}}$, $\varepsilon_{\mathbf{n}}s_{\mathbf{n}} = s_{\mathbf{n}}$.

Hence the discrete function $\mathbf{\Psi}^s_\beta(p^s)$ in Eq. (40) can be rewritten as a sum of continuous functions:

$$d^s \sum_{p^s} \mathbf{\Psi}^s_\beta(p^s)\delta(y - p^s d^s) = \sum_{n=1}^{\infty} \begin{pmatrix} s_{\mathbf{n}} \sin(k_{\mathbf{n}} y) \\ -\varepsilon_{\mathbf{n}} \cos(k_{\mathbf{n}} y) \\ s_{\mathbf{n}} \sin(k_{\mathbf{n}} y) \\ \varepsilon_{\mathbf{n}} \cos(k_{\mathbf{n}} y) \end{pmatrix}. \tag{B.11}$$

The effect on the shell, fixed on the edges, of the distributed moment $m^s_y(x,y)$ is equivalent to the effect of an additional distributed normal force $-\partial m^s_y(x,y)/\partial y$. Therefore the three components of stringer response to the shell forces $\mathbf{q}^s$ can be expressed through the discrete forces $\mathbf{q}^{\delta s}$ and the generalized forces $Q^{\delta s}_{\beta m}$ as

$$\mathbf{q}^s \equiv \begin{pmatrix} q^s_u \\ q^s_v \\ q^s_w \end{pmatrix} = \begin{pmatrix} q^s_u \\ q^s_v \\ q^{s1}_w - \partial m^s_y/\partial y \end{pmatrix}, \tag{B.12}$$

$$(q^s_u, q^s_v, q^{s1}_w, m^s_y R^{-1}) = \sum_{p^s} \mathbf{q}^{\delta s}(x, p^s)\delta(y - p^s d^s)$$

$$= d^s \sum_{\alpha,\beta} \sum_{m=1}^{\infty} \mathbf{Q}^{\delta s}_{\beta \mathbf{m}} \mathbf{\Phi}^s_{\mathbf{m}}(x) \sum_{p^s} \mathbf{\Psi}^s_\beta(p^s)\delta(y - p^s d^s). \tag{B.13}$$

Eqs. (32), (33), (39), (40), (B.11)–(B.13) give the relation between the vector of generalized stringer responses $\mathbf{Q}^s_{\mathbf{mn}}$ and the forces $\mathbf{Q}^{\delta s}_{\beta \mathbf{m}}$ through the $\mathbf{F}_{\mathbf{n}}$ matrix:

$$\mathbf{Q}^s_{\mathbf{mn}} = \mathbf{F}_{\mathbf{n}} \mathbf{Q}^{\delta s}_{\beta \mathbf{m}}, \quad \mathbf{F}_{\mathbf{n}} = \begin{pmatrix} s_{\mathbf{n}} & 0 & 0 & 0 \\ 0 & \varepsilon_{\mathbf{n}} & 0 & 0 \\ 0 & 0 & s_{\mathbf{n}} & \varepsilon_{\mathbf{n}} k_{\mathbf{n}} R \end{pmatrix}. \tag{B.14}$$

The relation between the generalized frame responses $\mathbf{Q}^r_{\mathbf{mn}}$ and forces $\mathbf{Q}^{\delta r}_{\alpha \mathbf{n}}$ acting on the frames can be obtained in the same way:

$$\mathbf{Q}^r_{\mathbf{mn}} = \mathbf{F}_{\mathbf{m}} \mathbf{Q}^{\delta r}_{\alpha \mathbf{n}}, \quad \mathbf{F}_{\mathbf{m}} = \begin{pmatrix} \varepsilon_{\mathbf{m}} & 0 & 0 & 0 \\ 0 & s_{\mathbf{m}} & 0 & 0 \\ 0 & 0 & s_{\mathbf{m}} & \varepsilon_{\mathbf{m}} k_{\mathbf{m}} R \end{pmatrix}. \tag{B.15}$$

Note that

$$\mathbf{F}_{\mathbf{n}} = \varepsilon_{\mathbf{n}} \mathbf{E}^{\mathrm{T}}_{\mathbf{n}}, \quad \mathbf{F}_{\mathbf{m}} = \varepsilon_{\mathbf{m}} \mathbf{E}^{\mathrm{T}}_{\mathbf{m}}. \tag{B.16}$$

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